Comment on "Diffusion in biased turbulence"

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(Received 3 December 2001; published 18 September 2002)

Recently Vlad *et al.* [Phys. Rev. E. **63**, 066304 (2001)] applied the method of decorrelation trajectories to the transport of tracers in stochastic velocity fields with constant drift, and found that the average Lagrangian velocity is smaller than the Eulerian average. As this contradicts a theoretical result due to Lumley [in *Mécanigue de la Turbulence*, International Conference of the CNRS, Marseille, 1961 (Centre National de la Recherche Scientifique, Paris, 1962)], two-dimensional numerical simulations are performed to confirm that the average Lagrangian and Eulerian velocities are in fact equal when the velocity field is divergence free.

DOI: 10.1103/PhysRevE.66.038301 PACS number(s): 47.27.Qb, 52.35.Ra, 02.50.Ey, 05.40.−a

In a recent paper [1], Vlad *et al.* considered transport by incompressible (divergence-free) stochastic velocity fields in the presence of a constant average drift. Results for the Lagrangian correlations, diffusion coefficents, and average Lagrangian velocity were derived using the method of decorrelation trajectories. In particular, it was claimed that the average Lagrangian velocity may differ from the Eulerian average V_d for large Kubo numbers when V_d is significantly smaller than the rms velocity of fluctuations. This result is contrary to a theorem of Lumley [2] (see also Ref. [3]) which states that the one-point, one-time probability distributions for the Eulerian and Lagrangian velocities are identical for homogeneous turbulence in an incompressible fluid. In this Comment we report on numerical simulations of transport in two-dimensional incompressible Gaussian fields which confirm Lumley's result and conclude that the average Lagrangian velocity equals the Eulerian average V_d, contradicting Ref. [1].

The dimensionless Langevin equation for transport of tracers is [see Ref. [1], Eq. (13)]

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{v}(\mathbf{x}(t),t)+\mathbf{V}_d, \tag{1}$$

$$\mathbf{x}(0)=\mathbf{0}, \tag{2}$$

where $\mathbf{x}(t)$ denotes the tracer position at time t. The random velocity field $\mathbf{v}(\mathbf{x},t)$ is assumed to be homogeneous and incompressible (divergence free), $\nabla\cdot\mathbf{v}=0$, and to have mean zero, $\langle\mathbf{v}\rangle=0$. Thus the average Eulerian velocity is equal to the constant vector $\mathbf{V}_d$. The Lagrangian velocity $\mathbf{V}$ is the velocity experienced by the tracers as they move along the random trajectories generated by Eq. (1) starting from $\mathbf{x}(t_0)=\mathbf{a}$:

$$\mathbf{V}(\mathbf{a},t)=\mathbf{v}(\mathbf{x}(t),t)+\mathbf{V}_d,$$

$$\mathbf{x}(t)=\mathbf{a}+\int_{t_0}^{t}\mathbf{V}(\mathbf{a},t')dt'.$$

The average Lagrangian velocity may differ from the Eulerian average if, for example, trapping effects are dominant. However, the conditions of homogeneity and incompressibility were shown by Lumley [2] to result in the equality of the Lagrangian and Eulerian average velocities $\langle\mathbf{V}\rangle=\mathbf{V}_d$. Here we briefly review Lumley's argument, following the presentation in Ref. [3]. First, Taylor expansion of the Lagrangian velocity in time leads to a formal series whose coefficients are all Eulerian quantities, and so to the immediate conclusion that Eulerian homogeneity implies Lagrangian homogeneity. We then choose a volume R of points and consider the integral

$$\int_R \mathbf{V}(\mathbf{a},t)d\mathbf{a}.$$

Because of incompressibility, the transform of variables from $\mathbf{a}$ to $\mathbf{x}$ has unit Jacobian, and so

$$\int_R \mathbf{V}(\mathbf{a},t)d\mathbf{a}=\int_{R'}[\mathbf{v}(\mathbf{x}(t),t)+\mathbf{V}_d]d\mathbf{x}, \tag{3}$$

where R' is the volume filled at time t by the fluid which filled volume R at time t_0. As the fields $\mathbf{v}$ and $\mathbf{V}$ are both homogeneous, the mean values of the functions under the integral signs do not depend on the coordinates. However, R' is a random volume which depends upon $\mathbf{v}(\mathbf{x},t)$ and so averaging under the integral signs is not immediately permitted. This difficulty is circumvented by the consideration of a sufficiently large initial volume R—in this manner, the relative error of replacing R' by R on the right-hand side of Eq. (3) can be made arbitrarily small. Averaging both sides of the resulting approximate equation and dividing by R yields the desired equality

$$\langle\mathbf{V}(\mathbf{a},t)\rangle=\langle\mathbf{v}(\mathbf{x},t)+\mathbf{V}_d\rangle=\mathbf{V}_d.$$

In other words, the average Lagrangian velocity $\langle\mathbf{V}\rangle$ equals the Eulerian average $\mathbf{V}_d$. Note that this theorem holds for both frozen and time-dependent velocities, is independent of the number of space dimensions, and that the condition of incompressibility (zero divergence) of the random field is a crucial element in the proof.

In order to demonstrate the application of Lumley's theorem to fields similiar to those employed in Ref. [1], and also to highlight the importance of the incompressibility condition, we perform numerical simulations of transport in frozen, two-dimensional, Gaussian velocity fields [4,5]. The av-

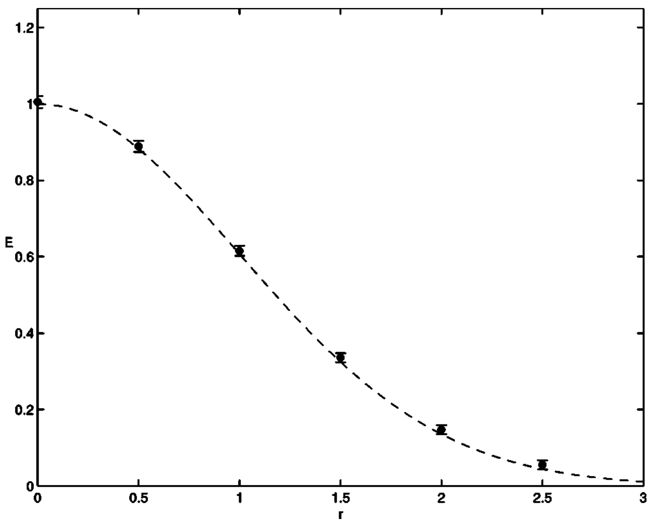

FIG. 1. Correlation function of the random velocity field as a function of separation distance $r=|\mathbf{r}|$. Symbols are the results of averaging over $N_r=16\,000$ realizations; the dashed line denotes the exact result $\exp(-r^2/2)$. The 95% confidence intervals are also shown.

erage Eulerian velocity $\mathbf{V}_d$ is taken to be in the x_1 direction, and the zero-mean random field $\mathbf{v}$ is generated in each realization by a sum of the form

$$\mathbf{v}(\mathbf{x})=\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\mathbf{z}_n\cos(\mathbf{k}_n\cdot\mathbf{x})+\mathbf{y}_n\sin(\mathbf{k}_n\cdot\mathbf{x}).$$

The components of each random vector $\mathbf{k}_n$ are chosen from independent Gaussian distributions of zero mean and unit variance. To ensure incompressibility, we first generate random vectors $\mathbf{a}_n$ and $\mathbf{b}_n$ which also have independent

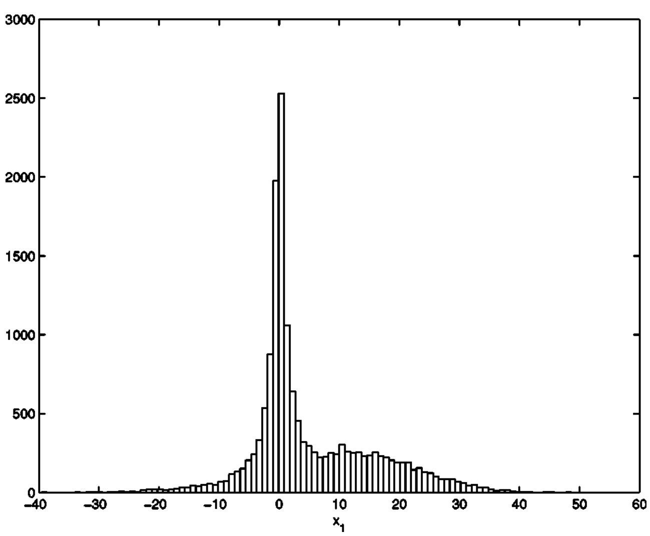

FIG. 2. Histogram of x_1 tracer positions at time $t=50$. The average Eulerian velocity is $V_d=0.1$ and the average displacement in the x_1 direction is 5.07 ± 0.17.

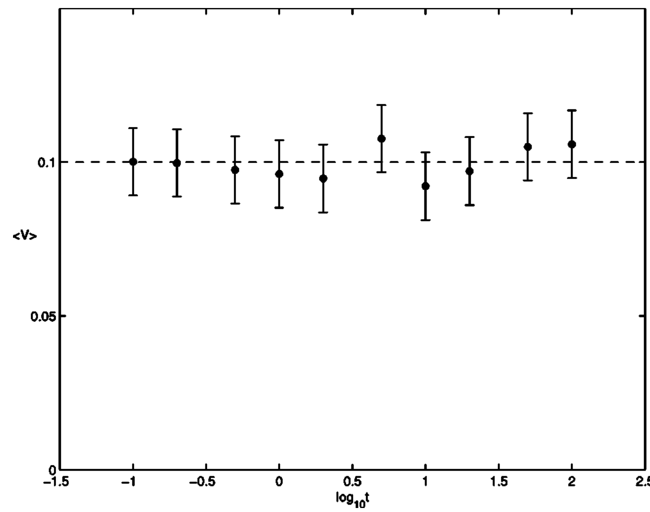

FIG. 3. Average Lagrangian velocity as a function of time (log scale). The dashed line represents the Eulerian average $V_d=0.1$.

Gaussian-distributed components of zero mean and unit variance. The amplitude vectors $\mathbf{z}_n$ and $\mathbf{y}_n$ are then calculated by setting

$$\mathbf{z}_n=\mathbf{a}_n-\frac{\mathbf{a}_n\cdot\mathbf{k}_n}{\mathbf{k}_n\cdot\mathbf{k}_n}\mathbf{k}_n,$$

$$\mathbf{y}_n=\mathbf{b}_n-\frac{\mathbf{b}_n\cdot\mathbf{k}_n}{\mathbf{k}_n\cdot\mathbf{k}_n}\mathbf{k}_n, \tag{4}$$

so that $\mathbf{z}_n\cdot\mathbf{k}_n=\mathbf{y}_n\cdot\mathbf{k}_n=0$, and therefore $\boldsymbol{\nabla}\cdot\mathbf{v}=0$. In the limit $N\to\infty$, the velocity field is Gaussian and homogeneous, with Eulerian correlation [compare Eq. (27) of Ref. [1]]

$$E(r)=\langle\mathbf{v}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x}+\mathbf{r})\rangle=\exp\left(-\frac{r^2}{2}\right),$$

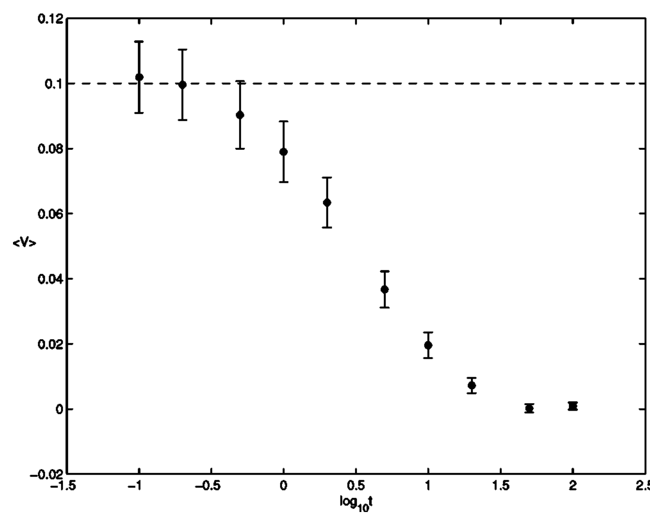

FIG. 4. Average Lagrangian velocity as a function of time (log scale) in a compressible velocity field. The dashed line represents the Eulerian average $V_d=0.1$.

where $r=|\mathbf{r}|$. This exact result is plotted for comparison with the numerical correlation in Fig. 1. The error bars denote the 95% confidence intervals. The number of modes is taken to be $N=100$ in all our experiments, with ensemble averages calculated over $N_r=16\,000$ realizations.

Having generated the random velocity field $\mathbf{v}(\mathbf{x})$ in each realization, the ordinary differential equation

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{v}(\mathbf{x}(t))+\mathbf{V}_d$$

with initial condition $\mathbf{x}(0)=\mathbf{0}$ is solved using standard numerical methods, and the tracer positions $\mathbf{x}(t)$ and Lagrangian velocities $\mathbf{v}(\mathbf{x}(t))+\mathbf{V}_d$ are stored at various times up to $t=100$. The average Eulerian x_1 velocity is taken to be $V_d=0.1$ for definiteness. The effects of trapping are evident in the histogram of the tracer displacements in the x_1 direction at time $t=50$ (Fig. 2), with a large number of tracers remaining close to the origin, but significant numbers escaping to ensure that the average displacement is $\langle x_1(50)\rangle=5.07\pm0.17$ at the 95% confidence level. The average Lagrangian velocity in the x_1 direction is plotted against $\log_{10}t$ in Fig. 3. We note that (in accordance with Lumley's theorem) the Lagrangian average does not display the significant deviations from the Eulerian average predicted by Vlad *et al.* [1], see their Figs. 2(a) and 3.

Figure 4 shows the dramatic effect of easing the incompressibility constraint, so that the velocity field is no longer divergence free. We replace Eq. (4) by $\mathbf{z}_n=\mathbf{a}_n/\sqrt{2}$, $\mathbf{y}_n=\mathbf{b}_n/\sqrt{2}$, so the condition $\nabla\cdot\mathbf{v}=0$ no longer holds. Proceeding as before, we find that the average Lagrangian velocity decreases from the Eulerian average velocity towards zero as time increases. Contrary to the situation in an incompressible velocity field, trapping of tracers at points of zero velocity is now possible, see Fig. 4.1 I of Ref. [6] and Ref. [7]. The number of trapped tracers increases with time, thus leading to the slowing of the mean Lagrangian velocity exhibited in Fig. 4. Indeed, we find that less than 4% of tracers have x_1-velocity magnitude of above 0.01 by time $t=100$, despite the Eulerian average being $V_d=0.1$.

In summary, we have shown that numerical simulation of tracer transport in two-dimensional frozen velocity fields agrees with the theoretical results of Lumley (and contradicts Ref. [1]), i.e., the Lagrangian and Eulerian average velocities are equal, provided the velocity is homogeneous and divergence free.

Support from Enterprise Ireland's International Collaboration scheme, and from the Institute of Nonlinear Science and the Faculty of Arts Research Fund, University College Cork is gratefully acknowledged.

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